arXiv:0811.4020v1 [gr-qc] 25 Nov 2008

The Equations of Motion in General Relativity of a Small Charged Black Hole

T. Futamase [*], P. A. Hogan [†] and Y. Itoh [‡]

Astronomical Institute, Graduate School of Science,

Tohoku University, Sendai 980-8578, Japan

PACS numbers: 04.20.-q,04.25.-g,04.25.Nx,04.40.-b

Abstract

We present the details of a model in General Relativity of a small charged black hole moving in an external gravitational and electromagnetic field. The importance of our model lies in the fact that we can derive the equations of motion of the black hole from the Einstein–Maxwell vacuum field equations *without encountering infinities*. The key assumptions which we base our results upon are (a) the black hole is isolated and (b) near the black hole the wave fronts of the radiation generated by its motion are smoothly deformed spheres. The equations of motion which emerge fit the pattern of the original De Witt and Brehme equations of motion (after they "renormalise"). Our calculations are carried out in a coordinate system in which the null hypersurface histories of the wave fronts can be specified in a simple way, with the result that we obtain a new explicit form, particular to our model, for the well–known "tail term" in the equations of motion.

[*]E-mail : tof@astr.tohoku.ac.jp

[†]E-mail : peter.hogan@ucd.ie

[‡]E-mail : yousuke@astr.tohoku.ac.jp

1 Introduction

The purpose of this paper is to present a model in General Relativity of a small charged black hole moving in an external gravitational and electromagnetic field. The external field is a solution of the Einstein–Maxwell field equations and the history of the black hole is a non–singular time–like world–line in this background space–time. The presence of the black hole is envisaged as a small perturbation of the background space–time which is singular on this world–line ($r = 0$). As the world–line is approached (by letting $r \to 0$) and the mass m and charge e of the black hole are taken to be small, such that in the limit $m \to 0$ and $e \to 0$ the ratios e/r and m/r remain finite, the perturbed gravitational field (Weyl tensor) is predominantly that of the Reissner–Nordstrom black hole and the perturbed electromagnetic field (Maxwell tensor) is predominantly the Coulomb field and the calculations described in this paper confirm that these limits can be achieved. These requirements guide us in our choice of expansions, in integer powers of r, of functions appearing in the metric tensor of the space–time and the potential 1–form.

We take the view that the assumed expansions restrict the model that we are constructing of a small charged black hole moving in external electromagnetic and gravitational fields. Their generality is a topic of further study. Assuming that the moving small black hole is isolated (in particular that there are no singular null geodesic generators of the histories of the wave fronts produced by its motion) we demonstrate how its approximate equations of motion are derived from the field equations as the requirement that the wave–fronts emerging from the moving black hole are, in the neighbourhood of the black hole, smoothly deformed 2–spheres. Approximations are based solely on the smallness of the mass and charge of the black hole (in particular slow motion is not assumed). The equations of motion derived fit the pattern of that of the original De Witt and Brehme [1] equations after they have "renormalised". Among the significant features of our approach is (a) the use of a coordinate system attached to the null hypersurface histories of the wave–fronts of the radiation produced by the black hole motion, (b) the emergence of the approximate equations of motion from the requirement that the wave–fronts are smoothly deformed 2–spheres near the black hole, (c) the explicitness of the so–called tail term specific to our model and (d) the absence of infinities arising in our process.

The topic of this paper has been an active area of research in general relativity beginning with the De Witt and Brehme [1] work (motivated by the classic paper by Dirac [2]). The equations of motion have been derived also by Beig [3] and by Barut and Villarroel [4]. For important recent work

on the equations of motion with radiation reaction of small black holes see [5, 6, 7, 8]. Some of the latter has centered on the problem of identifying tensor fields in the vicinity of the black hole which are singular or non–singular, as the case may be, on the world–line of the small black hole in the background space–time and the associated "regularization procedures" (see, for example [11]–[18] and the review [8]). Our work is complementary to the studies listed above. The precise relationship to them is a topic of future study however. Nevertheless recent work which should particularly be compared with our approach is section 4 in [5], section 19 in [9] and [10], although the latter applies only to vacuum background geometries. Earlier work that has more in common with our approach, but is more specialized, can be found in for example [19] (where the external field is considered weak) and in [20], [21] and [22] (where the space–time is less general than here).

The outline of the paper is as follows: In section 2 the Einstein–Maxwell background space–time is described in a suitable coordinate system for our purposes and some consequences of the field equations required later are derived. The charged black hole space–time is introduced as a perturbation of the background space–time in section 3 and the consequences of imposing, approximately, the vacuum Einstein–Maxwell field equations are given. In this section the method of extracting the equations of motion of the black hole, using the field equations and the properties of the wave–fronts near the black hole, is described and the equations of motion are derived. Section 4 is a brief discussion highlighting some properties of the equations of motion derived in section 3. To make the paper as self–contained as possible calculations required for section 2 are listed in Appendix A while section 3 requires both the calculations listed in Appendix A and the more extensive ones listed in Appendix B.

2 The Background Space–Time

We consider a small charged black hole moving in external gravitational and electromagnetic fields. We model the external fields by a potential 1–form and a space–time manifold on which it is defined which are solutions of the vacuum Einstein–Maxwell field equations. This space–time, which is otherwise unspecified in this work, contains a time–like world line ($r = 0$) on which the background Maxwell field (the Maxwell tensor field) and the background gravitational field (the Weyl tensor field) are non–singular. In the next section the small charged black hole, with mass m and charge e, is introduced as a perturbation of this space–time which is singular on the world line $r = 0$. The perturbed space–time will be an approximate solution

of the vacuum Einstein–Maxwell field equations having the property that in the limits $e \to 0$, $m \to 0$ and $r \to 0$ such that the ratios e/r and m/r remain finite the Maxwell field is dominated by the Coulomb field of the charge and the gravitational field (Weyl tensor field) is dominated by the Reissner–Nordstrom field of a charged black hole. To make all of these requirements more specific we begin by writing the line–element of the background in a coordinate system attached to a family of null hypersurfaces in the space–time having in general shear and expansion [23]. The line–element, in terms of a convenient basis of 1–forms, reads:

$$ds^2 = -(\vartheta^1)^2 - (\vartheta^2)^2 + 2\,\vartheta^3\,\vartheta^4 \ , \tag{2.1}$$

where

$$\begin{aligned}
\vartheta^1 &= r\,p^{-1}(e^{\alpha}\cosh\beta\,dx + e^{-\alpha}\sinh\beta\,dy + a\,du) \ , && (2.2)\\
\vartheta^2 &= r\,p^{-1}(e^{\alpha}\sinh\beta\,dx + e^{-\alpha}\cosh\beta\,dy + b\,du) \ , && (2.3)\\
\vartheta^3 &= dr + \frac{c}{2}\,du \ , && (2.4)\\
\vartheta^4 &= du \ . && (2.5)
\end{aligned}$$

A derivation of this line–element can be found in [24]. It is completely general, containing six functions $p,\ \alpha,\ \beta,\ a,\ b,\ c$ of the four coordinates x, y, r, u. It is therefore equivalent to line–elements constructed by Sachs [25] and by Newman and Unti [26]. The particular form we consider here was designed to allow the Robinson–Trautman [27][28] line–elements to emerge as a convenient special case (this case corresponds to putting $\alpha = \beta = 0$) and this is also a reason why the form is useful for the subject of the present paper. The hypersurfaces $u =$ constant are null and are generated by the null geodesic integral curves of the vector field $\partial/\partial r$. The coordinate r is an affine parameter along these curves and these null geodesics have (complex) shear

$$\sigma = \frac{\partial\alpha}{\partial r}\,\cosh 2\beta + i\,\frac{\partial\beta}{\partial r} \ , \tag{2.6}$$

and (real) expansion

$$\rho = \frac{\partial}{\partial r}\log(r\,p^{-1}) \ . \tag{2.7}$$

We take this "background" space–time to be a solution of the vacuum Einstein–Maxwell field equations. We take the history of the small charged black hole to be a (non–singular) time–like world line in this space–time and we take this world–line to have equation $r = 0$. Assuming the line–element to be regular on and in the neighbourhood of $r = 0$ we expand the six functions

introduced above in powers of r as follows (the choice of initial terms here is dictated by the classical work of Fermi [29] mentioned below):

$$\begin{aligned} p &= P_0(1 + q_2\, r^2 + q_3\, r^3 + \dots) \ , && (2.8)\\ \alpha &= \alpha_2\, r^2 + \alpha_3\, r^3 + \dots \ , && (2.9)\\ \beta &= \beta_2\, r^2 + \beta_3\, r^3 + \dots \ , && (2.10)\\ a &= a_1\, r + a_2\, r^2 + \dots \ , && (2.11)\\ b &= b_1\, r + b_2\, r^2 + \dots \ , && (2.12)\\ c &= c_0 + c_1\, r + c_2\, r^2 + \dots \ . && (2.13)\end{aligned}$$

In the coordinates (x, y, r, u) the potential 1–form of the background electromagnetic field takes the form

$$A = L\, dx + M\, dy + K\, du \ . \tag{2.14}$$

In the neighbourhood of the world line $r = 0$ we expand the coefficients of the differentials in (2.14) in positive powers of r in such a way that the corresponding electromagnetic field is non–singular on $r = 0$ (since this is the *external* field). A study of the exterior derivative of the 1–form (2.14) reveals that the appropriate expansions of the coefficients of the differentials are:

$$\begin{aligned} L &= r^2 L_2 + r^3 L_3 + \dots \ , && (2.15)\\ M &= r^2 M_2 + r^3 M_3 + \dots \ , && (2.16)\\ K &= r\, K_1 + r^2 K_2 + \dots \ . && (2.17)\end{aligned}$$

The functions appearing here and in (2.8)–(2.13) as coefficients of the different powers of r are real–valued functions of x, y, u only. We take the coordinates x, y, u in the range $(-\infty, +\infty)$ and r in the range $[0, +\infty)$. Had we included in (2.14) a term $W\, dr$ we would have had to take $W = W_1 r + W_2 r^2 + \dots$, and this can be removed by adding a gauge term to (2.14) without changing the form of the expansions (2.15)–(2.17).

It is easily seen from (2.6) and (2.7) that the complex shear and expansion of the integral curves of $\partial/\partial r$ are now given respectively by

$$\begin{aligned} \sigma &= 2\,(\alpha_2 + i\beta_2)\, r + 3\,(\alpha_3 + i\beta_3)\, r^2 + \dots \ , && (2.18)\\ \rho &= \frac{1}{r} - 2\, q_2\, r - 3\, q_3\, r^2 + \dots \ . && (2.19)\end{aligned}$$

Thus near $r = 0$ (for small values of r) the null hypersurfaces $u = \text{constant}$ resemble future null cones with vertices on the world line $r = 0$. Following

the classical work of Fermi [29] (see also [30, 31]) we can, without loss of generality, take the metric tensor in the neighbourhood of the world line $r=0$ to be the Minkowskian metric tensor, when convenient in rectangular Cartesian coordinates and time, up to terms of order r^2. This has led us to the starting terms chosen in (2.8)–(2.13). In addition in (2.8) and (2.13) we can write (see, for example [26] and Appendix A)

$$P_0 = x\,v^1 + y\,v^2 + \left\{1-\frac{1}{4}(x^2+y^2)\right\}\,v^3 + \left\{1+\frac{1}{4}(x^2+y^2)\right\}\,v^4\ , \qquad (2.20)$$

and

$$c_0 = 1 = \Delta\,\log P_0\ , \qquad \Delta = P_0^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\ , \qquad c_1 = -2\,h_0\ , \qquad (2.21)$$

with

$$h_0 = \eta_{ij}\,a^i\,k^j = \frac{\partial}{\partial u}(\log P_0)\ , \qquad (2.22)$$

$$-P_0\,k^i = x\delta^i_1 + y\delta^i_2 + \left\{1-\frac{1}{4}(x^2+y^2)\right\}\delta^i_3 - \left\{1+\frac{1}{4}(x^2+y^2)\right\}\delta^i_4\ . \qquad (2.23)$$

In these formulas $v^i(u)$ is the 4–velocity of the particle with world line $r=0$ calculated in rectangular Cartesian coordinates and time (X^i) (Latin indices take values 1, 2, 3, 4) and u is arc length or proper–time along this line. $\eta_{ij}=\mathrm{diag}(-1,-1,-1,+1)$ are the components of the Minkowskian metric tensor in coordinates (X^i) and thus $\eta_{ij}\,v^i\,v^j=+1$. Also $a^i=dv^i/du$ is the 4–acceleration of the particle with world–line $r=0$ (hence $\eta_{ij}\,v^i\,a^j=0$) and on $r=0$ we have $k^i\,\partial/\partial X^i=\partial/\partial r$ with $\eta_{ij}k^i\,v^j=1$ (this latter equation applied to (2.23) yields (2.20)). The operator Δ in (2.21) is the Laplacian on the unit 2–sphere.

The relationship between the rectangular Cartesian coordinates and time (X^i) and the coordinates (x,y,r,u) near the world line $r=0$ is given by (see, for example, [32])

$$X^i = x^i(u) + r\,k^i\ , \qquad (2.24)$$

neglecting $O(r^2)$–terms. Thus the world line $r=0$ has parametric equations $X^i=x^i(u)$ and the unit tangent vector or 4–velocity vector introduced above has components $v^i=dx^i/du$. We list in Appendix A some well known useful formulas arising from (2.24) which we will refer to in the sequel.

The electromagnetic field calculated with the 1–form (2.14) has the form

$$F = dA = \frac{1}{2}F_{ab}\,\vartheta^a\wedge\vartheta^b\ , \qquad (2.25)$$

with $F_{ab} = -F_{ba}$ and ϑ^a given by (2.2)–(2.5). Imposing Maxwell's vacuum field equations $d^*F = 0$, where the star denotes the Hodge dual, we find in leading order (in powers of the coordinate r) the following equations to be satisfied by the functions $L_2,\ M_2,\ K_1$:

$$K_1 = P_0^2\left(\frac{\partial L_2}{\partial x} + \frac{\partial M_2}{\partial y}\right) \ , \qquad \Delta K_1 + 2\,P_0^2\left(\frac{\partial L_2}{\partial x} + \frac{\partial M_2}{\partial y}\right) = 0 \ , \tag{2.26}$$

and

$$\frac{\partial K_1}{\partial x} + 2\,L_2 - \frac{\partial}{\partial y}\left\{P_0^2\left(\frac{\partial M_2}{\partial x} - \frac{\partial L_2}{\partial y}\right)\right\} = 0 \ , \tag{2.27}$$

$$\frac{\partial K_1}{\partial y} + 2\,M_2 + \frac{\partial}{\partial x}\left\{P_0^2\left(\frac{\partial M_2}{\partial x} - \frac{\partial L_2}{\partial y}\right)\right\} = 0 \ . \tag{2.28}$$

We see immediately that the second of (2.26) is a consequence of (2.27) and (2.28). Also (2.26) implies that K_1 is an $l = 1$ spherical harmonic:

$$\Delta K_1 + 2\,K_1 = 0 \ . \tag{2.29}$$

For our purposes the only functions of x, y, u appearing in (2.15)–(2.17) that we will require in the sequel are L_2, M_2 and K_1 and so we need not pursue Maxwell's vacuum field equations to higher order in powers of r.

The leading terms in the tetrad components of the Maxwell field F_{ab} that we will require are given by

$$F_{13} = -2\,P_0\,L_2 + O(r) \ , F_{23} = -2\,P_0\,M_2 + O(r) \ , F_{34} = K_1 + O(r) \ . \tag{2.30}$$

On $r = 0$ we can replace the basis 1–forms (2.2)–(2.5) by their Minkowskian counterparts (A.11)–(A.14) and the coordinates x, y, r, u by the coordinates (X^i). Thus evaluating (2.30) on $r = 0$ yields

$$L_2 = \frac{1}{2}F_{ij}(u)\,k^i\,\frac{\partial k^j}{\partial x} \ , \qquad M_2 = \frac{1}{2}F_{ij}(u)\,k^i\,\frac{\partial k^j}{\partial y} \ , \qquad K_1 = F_{ij}\,k^i\,v^j \ . \tag{2.31}$$

where $F_{ij}(u) = -F_{ji}(u)$ are the components of the (external) Maxwell field calculated in the coordinates (X^i) on $r = 0$. We can now verify directly that these expressions for $L_2,\ M_2,\ K_1$ satisfy the Maxwell equations (2.26)–(2.29) above. For example on substituting L_2, M_2 from (2.31) into first of (2.26) yields

$$K_1 = P_0^2\left(\frac{\partial L_2}{\partial x} + \frac{\partial M_2}{\partial y}\right) = \frac{1}{2}F_{ij}(u)k^i\,\Delta k^j = F_{ij}(u)\,k^i\,v^j \ , \tag{2.32}$$

with the last equality following from (A.15) and (A.16) added together.

Taking the electromagnetic field above as source, Einstein's field equations for the background space–time read

$$R_{ab} = 2\, E_{ab}\ , \tag{2.33}$$

where R_{ab} are the components of the Ricci tensor calculated on the tetrad given via the 1–forms (2.2)–(2.5). E_{ab} are the tetrad components of the electromagnetic energy–momentum tensor calculated using the Maxwell tensor (2.25) according to the formula

$$E_{ab} = F_{ca}\, F^{c}{}_{b} - \frac{1}{4}\, g_{ab}\, F_{cd}\, F^{cd}\ . \tag{2.34}$$

Here g_{ab} are the tetrad components of the metric tensor and indices on F_{ab} are raised using its inverse. We will give here only the consequences of the field equations (2.33) near $r = 0$ which will be useful later. To satisfy $R_{33} = 2\, E_{33} + O(r)$ we must have

$$q_2 = \frac{2}{3}\, P_0^2 (L_2^2 + M_2^2)\ , \tag{2.35}$$

for q_2 appearing in (2.8). With L_2, M_2 given by (2.31), and using (A.8) this can be written

$$q_2 = -\frac{1}{6}\, F^{p}{}_{i}\, F_{pj}\, k^i\, k^j\ . \tag{2.36}$$

Here $F_{pj} = F_{pj}(u)$ (and $F^{p}{}_{i} = \eta^{pq} F_{qi}(u)$) is the Maxwell tensor, in coordinates (X^i) evaluated on $r = 0$. Now to have

$$R_{12} = 2\, E_{12} + O(r),\ R_{11} - R_{22} = 2\, (E_{11} - E_{22}) + O(r)\ , \tag{2.37}$$

requires

$$2\, (\alpha_2 + i\beta_2) = -\frac{\partial}{\partial\bar{\zeta}} \left(a_1 + ib_1 + 4\, P_0^2 \frac{\partial q_2}{\partial\bar{\zeta}} \right)\ , \tag{2.38}$$

where $\zeta = x + iy$ and a bar will denote complex conjugation. Next, for $A = 1, 2$,

$$R_{A3} = 2\, E_{A3} + O(r)\ , \tag{2.39}$$

provided

$$a_1 + ib_1 + 4\, P_0^2 \frac{\partial q_2}{\partial\bar{\zeta}} = 2\, P_0^4 \frac{\partial}{\partial\zeta} (P_0^{-2} (\alpha_2 + i\beta_2))\ . \tag{2.40}$$

Putting (2.38) and (2.40) together we arrive at

$$\frac{\partial}{\partial\bar{\zeta}} \left\{ P_0^4 \frac{\partial}{\partial\zeta} (P_0^{-2} (\alpha_2 + i\beta_2)) \right\} = -(\alpha_2 + i\beta_2)\ . \tag{2.41}$$

The approximate field equations $R_{AA} = 2\,E_{AA} + O(r)$ and $R_{34} = 2\,E_{34} + O(r)$ both yield the same equation for the function c_2 appearing in (2.13) which will not be used in the sequel. The remaining field equations in the approximate form

$$R_{14} + iR_{24} = 2\,(E_{14} + iE_{24}) + O(r),\ \ R_{44} = 2\,E_{44} + O(r)\ , \tag{2.42}$$

are automatically satisfied. To check this requires some lengthy calculations. With the Einstein–Maxwell vacuum field equations satisfied in the neighbourhood of the world–line $r = 0$ we find the following tetrad components C_{abcd} of the Weyl conformal curvature tensor in the neighbourhood of $r = 0$:

$$C_{1313} + iC_{1323} = 6\,(\alpha_2 + i\beta_2) + O(r)\ , \tag{2.43}$$

$$C_{3431} + iC_{3432} = \frac{3}{2}P_0^{-1}\left(a_1 + ib_1 + 4\,P_0^2\frac{\partial q_2}{\partial\bar{\zeta}}\right) + O(r)\ . \tag{2.44}$$

Using the 1–forms defined in (A.11)–(A.14) and q_2 given by (2.36) we deduce from these that

$$\alpha_2 = \frac{1}{6}P_0^2 C_{ijkl}\,k^i\,\frac{\partial k^j}{\partial x}\,k^k\,\frac{\partial k^l}{\partial x}\ , \tag{2.45}$$

$$\beta_2 = \frac{1}{6}P_0^2 C_{ijkl}\,k^i\,\frac{\partial k^j}{\partial x}\,k^k\,\frac{\partial k^l}{\partial y}\ , \tag{2.46}$$

$$a_1 = \frac{2}{3}P_0^2\left(C_{ijkl}\,k^i\,v^j\,k^k\,\frac{\partial k^l}{\partial x} + F^p{}_i\,F_{pj}\,k^i\,\frac{\partial k^j}{\partial x}\right)\ , \tag{2.47}$$

$$b_1 = \frac{2}{3}P_0^2\left(C_{ijkl}\,k^i\,v^j\,k^k\,\frac{\partial k^l}{\partial y} + F^p{}_i\,F_{pj}\,k^i\,\frac{\partial k^j}{\partial y}\right)\ . \tag{2.48}$$

Here $C_{ijkl}(u)$ are the components of the Weyl tensor of this "background" space–time calculated on $r = 0$ in the coordinates (X^i). Hence they satisfy $C_{ijkl} = -C_{jikl} = -C_{ijlk} = C_{klij}$ and $\eta^{il}\,C_{ijkl} = 0$. Among the coefficients of the powers of r in (2.8)–(2.13) we shall only require here the functions $q_2, \alpha_2, \beta_2, a_1, b_1$ given by (2.36) and (2.45)–(2.48), along with P_0, c_0 and c_1 given in (2.20) and (2.21). Using the derivatives of k^i listed in (A.15)–(A.17) one can directly verify that the field equations (2.38) and (2.40) (and thus (2.41)) are satisfied by (2.36) and (2.45)–(2.48).

3 Black Hole Perturbation of Background

We introduce the small charged black hole as a perturbation of the background Einstein–Maxwell space–time described above. With the mass m

and charge e considered small of first order we write $m = O_1$ and $e = O_1$. The perturbation is introduced by modifying the expansions (2.8)–(2.13) as follows:

$$\begin{aligned} p &= \hat{P}_0(1 + \hat{q}_2\, r^2 + \hat{q}_3\, r^3 \dots) \ , && (3.1)\\ \alpha &= \hat{\alpha}_2\, r^2 + \hat{\alpha}_3\, r^3 + \dots && (3.2)\\ \beta &= \hat{\beta}_2\, r^2 + \hat{\beta}_3\, r^3 + \dots && (3.3)\\ a &= \frac{\hat{a}_{-1}}{r} + \hat{a}_0 + \hat{a}_1\, r + \hat{a}_2\, r^2 + \dots \ , && (3.4)\\ b &= \frac{\hat{b}_{-1}}{r} + \hat{b}_0 + \hat{b}_1\, r + \hat{b}_2\, r^2 + \dots \ , && (3.5)\\ c &= \frac{e^2}{r^2} - \frac{2\,(m + 2\,\hat{f}_{-1})}{r} + \hat{c}_0 + \hat{c}_1\, r + \hat{c}_2\, r^2 \dots . && (3.6) \end{aligned}$$

The coefficients of the powers of r are functions of x, y, u. The hats are introduced to distinguish these coefficients from the background coefficients. Functions here that have non–zero background values are assumed to differ from their background values by O_1–terms. Functions which vanish in the background are assumed to have orders of magnitude: $\hat{a}_{-1} = O_1$, $\hat{a}_0 = O_1$, $\hat{b}_{-1} = O_1$, $\hat{b}_0 = O_1$, $\hat{f}_{-1} = O_2$. The perturbed potential 1–form is taken to have the form (2.14) with the expansions (2.15)–(2.17) modified to read:

$$\begin{aligned} L &= \hat{L}_0 + r^2 \hat{L}_2 + r^3 \hat{L}_3 + \dots \ , && (3.7)\\ M &= \hat{M}_0 + r^2 \hat{M}_2 + r^3 \hat{M}_3 + \dots \ , && (3.8)\\ K &= \frac{(e + \hat{K}_{-1})}{r} + \hat{K}_0 + r\, \hat{K}_1 + r^2 \hat{K}_2 + \dots \ . && (3.9) \end{aligned}$$

Again functions here which are non–zero in the background are assumed to differ from their background values by O_1–terms and those functions which vanish in the background are taken to have the orders of magnitude: $\hat{L}_0 = O_1$, $\hat{M}_0 = O_1$, $\hat{K}_{-1} = O_2$, $\hat{K}_0 = O_1$. All of these expansions have been chosen in order to satisfy the Reissner–Nordstrom black hole limit described in the Introduction. A careful study suggests that they are the minimal choice of expansions necessary to satisfy the black hole limit. We consider, however, that these assumptions restrict the model that we are constructing of a small black hole moving in external electromagnetic and gravitational fields and that their generalisation or otherwise is a topic for further study. In order to simplify the presentation we make the further assumption that near $r = 0$ the potential 1–form is predominantly the Liénard–Wiechert 1–form (given by $A = e\,(r^{-1} - h_0)\, du$ up to a gauge term). This is achieved with the further specialisation of the functions $\hat{L}_0, \hat{M}_0, \hat{K}_0$ to satisfy

$$\hat{L}_0 = O_2 \ , \ \hat{M}_0 = O_2 \ , \ \hat{K}_0 = -e\, h_0 + O_2 \ , \qquad (3.10)$$

with h_0 given by (2.22).

We emphasise that our approach is *not* one of "matched asymptotic expansions" in which so–called "near zone" and "far zone" expansions of field variables are matched in an intermediate "buffer zone". In our perturbed space–time each function appearing in the metric tensor and the potential 1–form has only one expansion in powers of r given by (3.1)–(3.6) and (3.7)–(3.9) and all of our deductions, including the equations of motion, are made using these expansions and the vacuum Einstein–Maxwell field equations.

The expansions we have chosen here ensure that the null–hypersurfaces u = constant in the perturbed space–time, which will play the role of the histories of the wave fronts of the radiation produced by the black hole motion, are approximately future null–cones for small r. Thus the wave fronts can be approximately 2–spheres near the black hole. Neglecting $O(r^4)$–terms the line–elements induced on these null hypersurfaces are given by

$$ds_0^2 = -r^2\hat{P}_0^{-2}\,(dx^2+dy^2) \qquad \text{with} \qquad \hat{P}_0 = P_0\,(1+Q_1+Q_2+O_3)\ , \tag{3.11}$$

with P_0 given by (2.20) and $Q_1 = O_1,\ Q_2 = O_2$. We shall require the perturbations of these 2–spheres, described by the functions Q_1 and Q_2 here, to be smooth so that as functions of x, y they are non–singular (in particular at $x = \pm\infty$ and/or $y = \pm\infty$) so that no "directional singularities" occur violating the notion of an isolated black hole. We will insist that in general in solving the Einstein–Maxwell vacuum field equations for the perturbed space–time and the perturbed electromagnetic field that such directional singularities are unacceptable. Then it will follow from the Einstein–Maxwell field equations that *necessary conditions for the 2–surfaces with line–elements (3.11) to be smooth, non–trivial deformations of 2–spheres will be the equations of motion of the black hole*. Such an origin of equations of motion can be found, for example, in [20], [21] and [22] (where the space–time is less general than here) and in [19] (where the external field is weak).

The left hand sides of the vacuum Einstein–Maxwell field equations for the perturbed space–time are now power series in r, each with a finite number of terms involving inverse powers of r and an infinite number of terms involving positive powers of r. The coefficients of the leading terms in each case are listed in Appendix B. They are functions of x, y, u and we require them to be small (in terms of e and m) with just sufficient accuracy to enable us to derive the functions in the expansions (3.1)–(3.6) and (3.7)–(3.9) required for the equations of motion of the black hole with an O_3–error (to include electromagnetic radiation reaction etc.). The errors that we tolerate to achieve this are indicated in Appendix B. Some of these errors could be reduced but with no influence on the equations of motion to the accuracy

that we calculate here. We begin by calculating the equations of motion with an O_2–error. In this case we require from Einstein‘s equations (for the moment requiring (B.33) and (B.39) only to be small of second order)

$$\hat{a}_{-1} = -4\, e\, P_0^2 L_2 + O_2 = O_1\ , \qquad \hat{b}_{-1} = -4\, e\, P_0^2 M_2 + O_2 = O_1\ , \tag{3.12}$$

and thus from Maxwell‘s equations (with (B.21) required at this stage to be small of first order) we have

$$\hat{K}_1 = P_0^2 \left(\frac{\partial L_2}{\partial x} + \frac{\partial M_2}{\partial y}\right) + O_1 = K_1 + O_1\ , \tag{3.13}$$

with K_1 given in (2.31). We also need

$$\hat{c}_0 = 1 + \Delta Q_1 + 2\, Q_1 + 8\, e\, F_{ij}\, k^i\, v^j + O_2\ , \tag{3.14}$$

which emerges from the requirement that (B.25) be small of second order. When these are substituted into the final Einstein field equation (given by requiring (B.52) for the moment to be small of second order) the differential equation that emerges for Q_1 is

$$-\frac{1}{2}\Delta(\Delta Q_1 + 2\, Q_1) = 6\, m\, a_i\, p^i - 6\, e\, F_{ij}\, p^i\, v^j + O_2\ . \tag{3.15}$$

Here $p^i = h^i_j\, k^j$ with $h^i_j = \delta^i_j - v^i\, v_j$ is the projection of k^i orthogonal to v^i and satisfies $p^i\, p_i = -1, p^i\, v_i = 0$ and since $p^i = k^i - v^i$ we see from (A.15) and (A.16) that $\Delta p^i + 2\, p^i = 0$ so that each component of p^i is an $l = 1$ spherical harmonic (a spherical harmonic Q of order l is a smooth solution of $\Delta Q + l(l+1)\, Q = 0$). Integrating (3.15), discarding the homogeneous solution with directional singularities, we have

$$\Delta Q_1 + 2\, Q_1 = 6\, m\, a_i\, p^i - 6\, e\, F_{ij}\, p^i\, v^j + A(u) + O_2\ , \tag{3.16}$$

where $A(u) = O_1$ is arbitrary. The first two terms on the right hand side will now lead to directional singularities in Q_1 unless $m\, a_i\, p^i - e\, F_{ij}\, p^i\, v^j = O_2$ for all p^i, which implies the equations of motion in first approximation:

$$m\, a_i = e\, F_{ij}\, v^j + O_2\ . \tag{3.17}$$

Hence we see the Lorentz 4–force due to the external electromagnetic field appearing. With (3.17) satisfied the equation (3.16) simplifies and the directional singularity free Q_1 is a linear combination of an $l = 1$ spherical harmonic (the smooth homogeneous solution) and an $l = 0$ spherical harmonic (the particular integral $A(u)/2$). In general perturbations of (3.11)

described by Q_1 or Q_2 which are $l = 0$ or $l = 1$ spherical harmonics are trivial. When the 2–surfaces with line–element (3.11) are viewed against the Euclidean 3–space background such terms in Q_1 or Q_2 merely infinitesimally change the radius of the sphere (if $l = 0$) or infinitesimally displace the origin of the sphere (if $l = 1$) . We shall consistently discard such "perturbations" which means that in this case we shall take $Q_1 = 0$.

From now on we take $Q_1 = 0$ in (3.11) and our objective is to solve the Einstein–Maxwell vacuum field equations for the perturbed space–time and the perturbed Maxwell field with sufficient accuracy to enable us derive the function Q_2 in (3.11) and the equations of motion with an O_3-error. The functions of x, y, u appearing in the potential 1–form (as coefficients in the expansions (3.7)–(3.9) that are involved are $\hat{L}_0$, $\hat{M}_0$, $\hat{K}_{-1}$ and the O_1–part of $\hat{K}_1$ and $\hat{K}_0$. Two of Maxwell's equations ((B.13) and (B.16) small of third order) read

$$\frac{\partial\hat{K}_{-1}}{\partial x} + 6\,e^2 L_2 + \frac{\partial}{\partial y}\left\{P_0^2\left(\frac{\partial\hat{M}_0}{\partial x} - \frac{\partial\hat{L}_0}{\partial y}\right)\right\} = O_3\ , \tag{3.18}$$

and

$$\frac{\partial\hat{K}_{-1}}{\partial y} + 6\,e^2 M_2 - \frac{\partial}{\partial x}\left\{P_0^2\left(\frac{\partial\hat{M}_0}{\partial x} - \frac{\partial\hat{L}_0}{\partial y}\right)\right\} = O_3\ . \tag{3.19}$$

These imply

$$\Delta\hat{K}_{-1} + 6\,e^2 F_{ij}\,k^i\,v^j = O_3\ , \tag{3.20}$$

which incidentally is (B.19), another of the Maxwell equations. This latter equation can immediately be integrated without introducing directional singularities to yield

$$\hat{K}_{-1} = 3\,e^2 F_{ij}\,k^i\,v^j + e^2 K(u) + O_3\ , \tag{3.21}$$

where $K(u) = O_0$ is an arbitrary function. From (3.18) and (3.19) we obtain

$$\Delta\left\{P_0^2\left(\frac{\partial\hat{M}_0}{\partial x} - \frac{\partial\hat{L}_0}{\partial y}\right)\right\} - 6\,e^2 P_0^2 F_{ij}\,\frac{\partial k^i}{\partial x}\,\frac{\partial k^j}{\partial y} = O_3\ . \tag{3.22}$$

Now the second term on the left here is an $l = 1$ spherical harmonic. In fact we can write

$$P_0^2 F_{ij}\,\frac{\partial k^i}{\partial x}\,\frac{\partial k^j}{\partial y} = F^*_{ij}\,k^i\,v^j\ , \tag{3.23}$$

where $F^*_{ij}(u)$ is the dual of $F_{ij}(u)$, on account of the relationship

$$\epsilon_{pqkl}\,k^p\,\frac{\partial k^q}{\partial x} = k_k\,\frac{\partial k_l}{\partial y} - k_l\,\frac{\partial k_k}{\partial y}\ , \tag{3.24}$$

where ϵ_{ijkl} is the Levi–Civita permutation symbol in four dimensions and k^i is given by (2.23). We integrate (3.22) to obtain

$$P_0^2\left(\frac{\partial\hat{M}_0}{\partial x}-\frac{\partial\hat{L}_0}{\partial y}\right)=-3\,e^2P_0^2F^*_{ij}\,k^i\,v^j+e^2S(u)+O_3\ , \tag{3.25}$$

with $S(u)=O_0$ arbitrary. This is the extent to which we shall require a knowledge of $\hat{L}_0$ and $\hat{M}_0$. Substituting (3.21) and (3.25) into the first order equations (3.18) and (3.19) verifies that they are now satisfied.We now look for the O_1–parts of $\hat{L}_2$ and $\hat{M}_2$. We shall write $\hat{L}_2=L_2+l_2+O_2$ and $\hat{M}_2=M_2+m_2+O_2$ with $l_2=O_1$ and $m_2=O_1$. We can obtain l_2 and m_2 indirectly as follows: in the light of (3.21) and (3.25) two of Einstein's field equations yield algebraically ((B.33) and (B.39) required to be O_3)

$$\begin{aligned}\hat{a}_{-1}&=-4\,e\,P_0^2(L_2+l_2)+6\,e^2P_0^2F^p{}_i\,F_{pj}\,k^i\,\frac{\partial k^j}{\partial x}+4\,e^2P_0^2M_2\,S(u)\\&\quad-4\,e^2P_0^2L_2\,K(u)+O_3\ ,\end{aligned} \tag{3.26}$$

$$\begin{aligned}\hat{b}_{-1}&=-4\,e\,P_0^2(M_2+m_2)+6\,e^2P_0^2F^p{}_i\,F_{pj}\,k^i\,\frac{\partial k^j}{\partial y}-4\,e^2P_0^2L_2\,S(u)\\&\quad-4\,e^2P_0^2M_2\,K(u)+O_3\ ,\end{aligned} \tag{3.27}$$

Define two functions $\hat{A}$ and $\hat{B}$ by

$$\hat{a}_{-1}+8\,P_0^2L_2\hat{K}_{-1}=-4\,e\,P_0^2L_2+\hat{A}\ , \tag{3.28}$$

and

$$\hat{b}_{-1}+8\,P_0^2M_2\hat{K}_{-1}=-4\,e\,P_0^2M_2+\hat{B}\ . \tag{3.29}$$

Now two of Einstein's field equations ((B.27) and (B.30) both O_3) can be written as equations for $\hat{A}$ and $\hat{B}$ neatly in the complex form

$$\frac{\partial}{\partial\bar{\zeta}}(\hat{A}+i\hat{B})=-8\,e^2P_0^2(L_2+iM_2)^2-2\,e^2(\alpha_2+i\beta_2)+O_3\ . \tag{3.30}$$

Here $L_2, M_2, \alpha_2, \beta_2$ are the background functions given by (2.31), (2.45) and (2.46). We note that $\alpha_2+i\beta_2$ can be expressed as a derivative with respect to $\bar{\zeta}$ given by (2.41). Similarly one can show that

$$P_0^2(L_2+iM_2)^2=-2\,\frac{\partial}{\partial\bar{\zeta}}\left(P_0^4(L_2+iM_2)\,\frac{\partial}{\partial\zeta}(L_2+iM_2)\right)\ . \tag{3.31}$$

Hence (3.30) can immediately be integrated giving a solution which is free of directional singularities:

$$\begin{aligned}\hat{A}+i\hat{B}&=2\,e^2P_0^4\frac{\partial}{\partial\zeta}\{P_0^{-2}(\alpha_2+i\beta_2)\}+16\,e^2P_0^4(L_2+iM_2)\,\frac{\partial}{\partial\zeta}(L_2+iM_2)\\&\quad+e^2U(u)+ie^2V(u)+O_3\ ,\end{aligned} \tag{3.32}$$

where $U(u) = O_0$ and $V(u) = O_0$ are functions of integration. Combining (3.26) with (3.28) and (3.27) with (3.29) and using the solution (3.32) means that we have now determined the functions l_2 and m_2 to be

$$\begin{aligned} l_2 &= e\,F_{pq}\,k^p\,\frac{\partial k^q}{\partial x}\,F_{ij}\,k^i\,v^j + \frac{1}{2}e\,F^p{}_i\,F_{pj}\,k^i\,\frac{\partial k^j}{\partial x} - \frac{1}{6}e\,C_{ijkl}\,k^i\,v^j\,k^k\,\frac{\partial k^l}{\partial x} \\ &\quad + \frac{1}{2}e\,S(u)\,F_{ij}\,k^i\,\frac{\partial k^j}{\partial y} + \frac{1}{2}e\,K(u)\,F_{ij}\,k^i\,\frac{\partial k^j}{\partial x} - \frac{1}{4}e\,P_0^{-2}U(u) + O_2\ , \end{aligned} \tag{3.33}$$

$$\begin{aligned} m_2 &= e\,F_{pq}\,k^p\,\frac{\partial k^q}{\partial y}\,F_{ij}\,k^i\,v^j + \frac{1}{2}e\,F^p{}_i\,F_{pj}\,k^i\,\frac{\partial k^j}{\partial y} - \frac{1}{6}e\,C_{ijkl}\,k^i\,v^j\,k^k\,\frac{\partial k^l}{\partial y} \\ &\quad - \frac{1}{2}e\,S(u)\,F_{ij}\,k^i\,\frac{\partial k^j}{\partial x} + \frac{1}{2}e\,K(u)\,F_{ij}\,k^i\,\frac{\partial k^j}{\partial y} - \frac{1}{4}e\,P_0^{-2}V(u) + O_2\ . \end{aligned} \tag{3.34}$$

One of Maxwell's equations ((B.21) required to be O_2) now gives us directly

$$\hat{K}_1 = F_{ij}\,k^i\,v^j + P_0^2\left(\frac{\partial l_2}{\partial x} + \frac{\partial m_2}{\partial y}\right) - 14\,e\,q_2 + O_2\ , \tag{3.35}$$

and we note that q_2 is given by (2.36).

We now turn our attention to the functions describing the perturbations of the metric tensor. We require $\hat{a}_0, \hat{b}_0, \hat{f}_{-1}$ and $\hat{c}_0$. Defining the variables

$$\hat{\mathcal{A}} = \hat{a}_0 + 4\,P_0^2 L_2\,\hat{K}_0\ , \qquad \text{and} \qquad \hat{\mathcal{B}} = \hat{b}_0 + 4\,P_0^2 M_2\,\hat{K}_0\ , \tag{3.36}$$

two of Einstein's equations ((B.28) and (B.31) taken to be O_2) can be written as the one complex equation

$$\frac{\partial}{\partial\bar{\zeta}}(\hat{\mathcal{A}} + i\hat{\mathcal{B}}) = 4\,m\,P_0^2(L_2 + iM_2)^2 + 4\,m\,(\alpha_2 + i\beta_2) + O_2\ . \tag{3.37}$$

This is integrated in similar fashion to (3.30). The solution should have the property that $P_0^{-1}\hat{a}_0$ and $P_0^{-1}\hat{b}_0$ are free of directional singularities since it is in this way that $\hat{a}_0$ and $\hat{b}_0$ appear in the metric tensor. The solutions of (3.37) therefore that we require are

$$\begin{aligned} \hat{a}_0 &= 2\,e\,P_0^2 a_p\,k^p\,F_{ij}\,k^i\,\frac{\partial k^j}{\partial x} - 2\,m\,P_0^2 F^p{}_i\,F_{pj}\,k^i\,\frac{\partial k^j}{\partial x} \\ &\quad -4\,m\,P_0^2 F_{pq}\,k^p\,v^q\,F_{ij}\,k^i\,\frac{\partial k^j}{\partial x} - \frac{4}{3}m\,P_0^2 C_{ijkl}\,k^i\,v^j\,k^k\,\frac{\partial k^l}{\partial x} \\ &\quad + X(u) + O_2\ , \end{aligned} \tag{3.38}$$

$$
\begin{aligned}
\hat{b}_0 &= 2\,e\,P_0^2 a_p\,k^p\,F_{ij}\,k^i\,\frac{\partial k^j}{\partial y} - 2\,m\,P_0^2 F^p{}_i\,F_{pj}\,k^i\,\frac{\partial k^j}{\partial y} \\
&\quad -4\,m\,P_0^2 F_{pq}\,k^p\,v^q\,F_{ij}\,k^i\,\frac{\partial k^j}{\partial y} - \frac{4}{3} m\,P_0^2 C_{ijkl}\,k^i\,v^j\,k^k\,\frac{\partial k^l}{\partial y} \\
&\quad +Y(u) + O_2\ , \qquad (3.39)
\end{aligned}
$$

where $X(u) = O_1$ and $Y(u) = O_1$ are functions of integration. We note that $\hat{a}_0$ and $\hat{b}_0$ possess directional singularities but $P_0^{-1}\hat{a}_0$ and $P_0^{-1}\hat{b}_0$ do not, as required. Another of Einstein's field equations ((B.50) required to be O_3) provides us with the equation

$$
\Delta \hat{f}_{-1} - 2\,e^2 a_i\,k^i + m\,P_0^2 \left(\frac{\partial}{\partial x}(P_0^{-2}\hat{a}_{-1}) + \frac{\partial}{\partial y}(P_0^{-2}\hat{b}_{-1}) \right) = O_3\ , \qquad (3.40)
$$

which using (3.26) and (3.27) can be rewritten as

$$
\Delta(\hat{f}_{-1} - e^2 a_i\,k^i - 2\,m\,e\,F_{ij}\,k^i\,v^j) = O_3\ , \qquad (3.41)
$$

from which we conclude that

$$
\hat{f}_{-1} = e^2 a_i\,k^i + 2\,m\,e\,F_{ij}\,k^i\,v^j + G(u) + O_3\ , \qquad (3.42)
$$

with $G(u) = O_2$ a function of integration. Now we obtain directly from an Einstein equation ((B.25) taken O_3) the function $\hat{c}_0$. After some considerable simplification it can be written

$$
\begin{aligned}
\hat{c}_0 &= 1 + 8\,e\,F_{ij}\,k^i\,v^j + \Delta Q_2 + 2\,Q_2 - 4\,e^2 C_{ijkl}\,k^i\,v^j\,k^k\,v^l \\
&\quad +8\,e^2 F^{ij}\,F_{ij} + 24\,e^2 (F_{ij}\,k^i\,v^j)^2 - 24\,e^2 F^p{}_i\,F_{pj}\,k^i\,v^j \\
&\quad +\frac{65}{3}\,e^2 F^p{}_i\,F_{pj}\,k^i\,k^j + 16\,e^2 K(u)\,F_{ij}\,k^i\,v^j + 4\,e^2 U(u)\,\frac{\partial}{\partial x}(\log P_0) \\
&\quad +4\,e^2 V(u)\,\frac{\partial}{\partial y}(\log P_0) + O_3\ . \qquad (3.43)
\end{aligned}
$$

The perturbation in the function $\hat{c}_1 = c_1 + O_1$ is now obtained from (B.26) or (B.49) required to be O_2. We shall not need this function in the sequel. The remaining quantities in (B.13)–(B.21) and (B.23)–(B.51) can now be evaluated and their orders of magnitude are indicated in Appendix B. The differential equation for Q_2 emerges from the remaining quantity (B.52) required to be O_3. This expression is lengthy and so we split it into a sum of three terms which we call T_1, T_2, T_3. The first of these terms involves only $\hat{a}_{-1}$ and $\hat{b}_{-1}$ and is given by

$$
T_1 = -\left(\frac{\partial \hat{a}_{-1}}{\partial x}\right)^2 - \left(\frac{\partial \hat{b}_{-1}}{\partial y}\right)^2 - \frac{1}{2}\left(\frac{\partial \hat{a}_{-1}}{\partial y}\right)^2 - \frac{1}{2}\left(\frac{\partial \hat{b}_{-1}}{\partial x}\right)^2
$$

$$
\begin{aligned}
&-\frac{\partial\hat a_{-1}}{\partial y}\,\frac{\partial\hat b_{-1}}{\partial x}+4\,\hat b_{-1}\,\frac{\partial\hat b_{-1}}{\partial y}\,P_0^{-1}\frac{\partial P_0}{\partial y}+4\,\hat a_{-1}\,\frac{\partial\hat a_{-1}}{\partial x}\,P_0^{-1}\frac{\partial P_0}{\partial x}\\
&+2\,\hat a_{-1}\frac{\partial\hat b_{-1}}{\partial y}\,P_0^{-1}\frac{\partial P_0}{\partial x}+2\,\hat b_{-1}\frac{\partial\hat a_{-1}}{\partial x}\,P_0^{-1}\frac{\partial P_0}{\partial y}+2\,\hat a_{-1}\,\frac{\partial\hat b_{-1}}{\partial x}\,P_0^{-1}\frac{\partial P_0}{\partial y}\\
&+2\,\hat b_{-1}\,\frac{\partial\hat a_{-1}}{\partial y}\,P_0^{-1}\frac{\partial P_0}{\partial x}-2\,(\hat a_{-1})^2P_0\,\frac{\partial^2}{\partial x^2}(P_0^{-1})-2\,(\hat b_{-1})^2P_0\,\frac{\partial^2}{\partial y^2}(P_0^{-1})\\
&-8\,\hat a_{-1}\,\hat b_{-1}\,P_0^{-2}\frac{\partial P_0}{\partial x}\,\frac{\partial P_0}{\partial y}-\hat a_{-1}\,\left\{\frac{\partial^2\hat a_{-1}}{\partial x^2}+\frac{\partial^2\hat b_{-1}}{\partial x\partial y}\right\}\\
&-\hat b_{-1}\,\left\{\frac{\partial^2\hat b_{-1}}{\partial y^2}+\frac{\partial^2\hat a_{-1}}{\partial x\partial y}\right\}\ .
\end{aligned}
\tag{3.44}
$$

To evaluate this we only require the leading terms in $\hat a_{-1}$ and $\hat b_{-1}$ given in (3.26) and (3.27). By (2.30) these can be written

$$
\hat a_{-1}=2\,e\,P_0F_{13}+O_2\ ,\qquad \text{and}\qquad \hat b_{-1}=2\,e\,P_0F_{23}+O_2\ . \tag{3.45}
$$

Direct calculation reveals

$$
\frac{\partial\hat a_{-1}}{\partial y}=-\frac{\partial\hat b_{-1}}{\partial x}+O_2=2e\left\{\frac{\partial P_0}{\partial y}\,F_{13}-\frac{\partial P_0}{\partial x}\,F_{23}+F_{12}\right\}+O_2\ , \tag{3.46}
$$

$$
\frac{\partial\hat a_{-1}}{\partial x}=\frac{\partial\hat b_{-1}}{\partial y}+O_2=2e\left\{\frac{\partial P_0}{\partial x}\,F_{13}+\frac{\partial P_0}{\partial y}\,F_{23}-F_{34}\right\}+O_2\ , \tag{3.47}
$$

$$
\begin{aligned}
\frac{\partial^2\hat a_{-1}}{\partial x\,\partial y}\ &=\ e\left\{-P_0^{-1}+2\,\frac{\partial^2P_0}{\partial y^2}-2\,P_0^{-1}\left(\frac{\partial P_0}{\partial x}\right)^2\right\}\,F_{23}-2\,e\,P_0^{-1}F_{24}\\
&\quad+2\,e\,P_0^{-1}\frac{\partial P_0}{\partial x}\,\frac{\partial P_0}{\partial y}\,F_{13}+2\,e\,P_0^{-1}\frac{\partial P_0}{\partial x}\,F_{12}-2\,e\,P_0^{-1}\frac{\partial P_0}{\partial y}\,F_{34}\ ,
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
\frac{\partial^2\hat b_{-1}}{\partial x\,\partial y}\ &=\ e\left\{-P_0^{-1}+2\,\frac{\partial^2P_0}{\partial x^2}-2\,P_0^{-1}\left(\frac{\partial P_0}{\partial y}\right)^2\right\}\,F_{13}-2\,e\,P_0^{-1}F_{14}\\
&\quad+2\,e\,P_0^{-1}\frac{\partial P_0}{\partial x}\,\frac{\partial P_0}{\partial y}\,F_{23}-2\,e\,P_0^{-1}\frac{\partial P_0}{\partial y}\,F_{12}-2\,e\,P_0^{-1}\frac{\partial P_0}{\partial x}\,F_{34}\ ,
\end{aligned}
\tag{3.49}
$$

where F_{ab} are the components of the external electromagnetic field evaluated on $r=0$ and referred to the tetrad given by (A.11)–(A.14). Substitution in (3.44) leads to the simplification

$$
T_1=4\,e^2\left\{(F_{13})^2+(F_{23})^2\right\}+8\,e^2\,\{F_{13}\,F_{14}+F_{23}\,F_{24}\}-8\,e^2(F_{34})^2+O_3\ . \tag{3.50}
$$

This is further reduced by noting that

$$(F_{13})^2 + (F_{23})^2 = -F^p{}_i\, F_{pj}\, k^i\, k^j \ , \tag{3.51}$$

$$F_{13}\, F_{14} + F_{23}\, F_{24} = -F^p{}_i\, F_{pj}\, k^i\, v^j + \frac{1}{2}\, F^p{}_i\, F_{pj}\, k^i\, k^j - (F_{ij} k^i\, v^j)^2 \ , \tag{3.52}$$

and

$$F_{34} = F_{ij} k^i\, v^j \ . \tag{3.53}$$

Hence we can write

$$T_1 = -8\, e^2 F^p{}_i\, F_{pj} k^i\, v^j - 16\, e^2 (F_{ij} k^i v^j)^2 \ . \tag{3.54}$$

The second term T_2 is chosen because it involves the function $\hat{c}_0$ explicitly. It is defined by

$$T_2 = \frac{1}{2}\, \hat{\Delta} \hat{c}_0 - \frac{1}{2} \hat{c}_0\, \hat{P}_0^2 \left(\frac{\partial}{\partial x} \left(\hat{P}_0^{-2} \hat{a}_{-1} \right) + \frac{\partial}{\partial y} \left(\hat{P}_0^{-2} \hat{b}_{-1} \right) \right) - \frac{1}{2} \hat{a}_{-1}\, \frac{\partial \hat{c}_0}{\partial x} - \frac{1}{2} \hat{b}_{-1}\, \frac{\partial \hat{c}_0}{\partial y} \ . \tag{3.55}$$

Explicit evaluation yields

$$\begin{aligned} T_2 \ = \ & -6\, F_{ij}\, k^i\, v^j + \frac{1}{2} \Delta (\Delta Q_2 + 2\, Q_2) - 3\, e^2 U(u)\, \frac{\partial}{\partial x} (\log P_0) \\ & -3\, e^2 V(u)\, \frac{\partial}{\partial y} (\log P_0) - 12\, e^2 K(u)\, F_{ij}\, k^i\, v^j + 11\, e^2 C_{ijkl}\, k^i\, v^j\, k^k\, v^l \\ & -\frac{59}{3}\, e^2 F^{ij}\, F_{ij} - 42\, e^2 (F_{ij}\, k^i\, v^j)^2 - 48\, e^2 F^p{}_i\, F_{pj}\, v^i\, v^j \\ & -59\, e^2 F^p{}_i\, F_{pj}\, k^i\, k^j + \frac{338}{3}\, e^2 F^p{}_i\, F_{pj}\, k^i\, v^j + O_3 \ . \end{aligned} \tag{3.56}$$

The final term T_3 consists of the terms remaining in (B.52). It is given by

$$\begin{aligned} T_3 \ = \ & 3\, m\, P_0^2 \left\{ \frac{\partial}{\partial x} (P_0^{-2} \hat{a}_0) + \frac{\partial}{\partial y} (P_0^{-2} \hat{b}_0) \right\} - 4\, \frac{\partial \hat{f}_{-1}}{\partial u} + 12\, \hat{f}_{-1}\, h_0 + 6\, m\, h_0 \\ & -4 e^2 P_0^2 L_2\, \frac{\partial \hat{K}_1}{\partial x} - 4 e^2 P_0^2 M_2\, \frac{\partial \hat{K}_1}{\partial y} - 2\, P_0^2 \left\{ \left(\frac{\partial \hat{K}_0}{\partial x} \right)^2 + \left(\frac{\partial \hat{K}_0}{\partial y} \right)^2 \right\} \\ & +8\, m\, P_0^2\, L_2\, \frac{\partial \hat{K}_0}{\partial x} + 8\, m\, P_0^2\, M_2\, \frac{\partial \hat{K}_0}{\partial y} - 4\, P_0^2 L_2 \frac{\partial \hat{K}_{-1}}{\partial x} - 4\, P_0^2 M_2 \frac{\partial \hat{K}_{-1}}{\partial y} \\ & -4\, P_0^2 \frac{\partial K_1}{\partial x}\, \frac{\partial \hat{K}_{-1}}{\partial x} - 4\, P_0^2 \frac{\partial K_1}{\partial y}\, \frac{\partial \hat{K}_{-1}}{\partial y} \\ & -\frac{5}{2} e^2\, P_0^2 \left\{ \frac{\partial}{\partial x} (P_0^{-2} a_1) + \frac{\partial}{\partial y} (P_0^{-2} b_1) \right\} \ . \end{aligned} \tag{3.57}$$

When this is evaluated with the functions derived above it results in (using $h_0 = a_i\, k^i$ again for convenience)

$$\begin{aligned}
T_3 \;\;=&\;\; 6\, m\, h_0 + 2\, e^2 a_i\, a^i + 14\, e^2 h_0^2 - 4\, e^2 \dot{h}_0 - 4\, \dot{G} + 12\, h_0\, G - 8\, m\, e\, \dot{F}_{ij}\, k^i\, v^j \\
&+ \left(6\, m^2 + \frac{5}{3}\, e^2\right)\, F^{ij}\, F_{ij} - (12\, m^2 + 5\, e^2)\, C_{ijkl}\, k^i\, v^j\, k^k\, v^l \\
&-6\, m\, X(u)\, \frac{\partial}{\partial x}(\log P_0) - 6\, m\, Y(u)\, \frac{\partial}{\partial y}(\log P_0) \\
&+(50\, e^2 - 36\, m^2)\, (F_{ij}\, k^i\, v^j)^2 + 12\, e^2 F^p{}_i\, F_{pj}\, v^i\, v^j \\
&- \left(36\, m^2 + \frac{14}{3}\, e^2\right)\, F^p{}_i\, F_{pj}\, k^i\, v^j + (18\, m^2 + 5\, e^2)\, F^p{}_i\, F_{pj}\, k^i\, k^j \\
&+O_3\ .
\end{aligned} \tag{3.58}$$

The approximate field equation giving us a differential equation for Q_2 is obtained from $T_1 + T_2 + T_3 = O_3$ with T_1, T_2, T_3 given by (3.54), (3.56) and (3.58). Making use of the unit space–like vector field p^i introduced prior to (3.16) we can write the result in the form

$$-\frac{1}{2}\Delta(\Delta Q_2 + 2\, Q_2) = A_0 + A_1 + A_2 + O_3\ , \tag{3.59}$$

where

$$A_0 = -4\, \dot{G} - \frac{16}{3} e^2 F^p{}_i\, F_{pj}\, v^i\, v^j\ , \tag{3.60}$$

which is an $l = 0$ spherical harmonic,

$$\begin{aligned}
A_1 \;\;=&\;\; 6\, m\, a_i\, p^i - 6\, e\, F_{ij}\, p^i\, v^j - 4\, e^2 h_i^k \dot{a}_k\, p^i - 8\, m\, e\, \dot{F}_{ij}\, p^i\, v^j \\
&-8\, e^2 h_i^k\, F^p{}_k\, F_{pj}\, p^i\, v^j + 12\, G\, a_i\, p^i - 12\, e^2 K(u)\, F_{ij}\, p^i\, v^j \\
&-3\, (e^2 U(u) + 2\, m\, X(u))\, \frac{\partial}{\partial x}(\log P_0) \\
&-3\, (e^2 V(u) + 2\, m\, Y(u))\, \frac{\partial}{\partial y}(\log P_0)\ ,
\end{aligned} \tag{3.61}$$

which is an $l = 1$ spherical harmonic, and

$$A_2 = 6\, (e^2 - 2m^2)\, \chi_{(1)} - 4\, (2\, e^2 + 9m^2)\, \chi_{(2)} + 18(m^2 - 3e^3)\, \chi_{(3)} + 18e^2 \chi_{(4)}\ , \tag{3.62}$$

which is an $l = 2$ spherical harmonic. Any $l = 2$ spherical harmonic can be written

$$\chi = D_{ij}(u)\, k^i\, k^j\ , \tag{3.63}$$

with

$$D_{ij} = D_{ji}\ , \qquad D_{ij}\, v^j = 0\ , \qquad \eta^{ij}\, D_{ij} = 0\ . \tag{3.64}$$

In (3.62) each $\chi_{(\alpha)}$ for $\alpha = 1, 2, 3, 4$ has the form

$$\chi_{(\alpha)} = {}_{(\alpha)}D_{ij}\, k^i\, k^j \ , \tag{3.65}$$

with

$$\begin{aligned}
{}_{(1)}D_{ij} &= C_{ikjl}\, v^k\, v^l \ , &(3.66)\\
{}_{(2)}D_{ij} &= F_{ik}v^k\, F_{jl}v^l - \frac{1}{3}h_{ij}F^p{}_k\, F_{pl}\, v^k\, v^l \ , &(3.67)\\
{}_{(3)}D_{ij} &= F^p{}_k\, F_{pl}\, h^k_i\, h^l_j - \frac{1}{3}h_{ij}\, h^{kl}\, F^p{}_k\, F_{pl} \ , &(3.68)\\
{}_{(4)}D_{ij} &= a_i\, a_j - \frac{1}{3}h_{ij}\, a_k\, a^k \ . &(3.69)
\end{aligned}$$

The last two terms in (3.61) can be simplified by noting that from (2.20) and (2.23) we can write

$$\frac{\partial}{\partial x}(\log P_0) = c_i\, k^i = c_i\, p^i \ , \qquad \frac{\partial}{\partial y}(\log P_0) = d_i\, k^i = d_i\, p^i \ , \tag{3.70}$$

with

$$c_i = \left(\frac{1}{2}(v^3 - v^4), 0, -\frac{1}{2}v^1, \frac{1}{2}v^1\right) \quad \text{and} \quad d_i = \left(0, \frac{1}{2}(v^3 - v^4), -\frac{1}{2}v^2, \frac{1}{2}v^2\right) \ , \tag{3.71}$$

covariant vectors (in coordinates (X^i)) defined along $r = 0$ and everywhere orthogonal to v^i. Provided $A_0 = O_3$, (3.59) can be integrated without the introduction of directional singularities to read

$$\Delta Q_2 + 2\, Q_2 = A_1 + \frac{1}{3}A_2 + O_3 \ , \tag{3.72}$$

up to the addition of an arbitrary function of u which makes a trivial contribution to Q_2. For Q_2 to be free of directional singularities we must have $A_1 = O_3$ for all p^i such that $p^i\, v_i = 0$ and this leads to the equations of motion

$$\begin{aligned}
m a_i &= e\, F_{ij}\, v^j + \frac{2}{3}e^2 h^k_i \dot{a}_k + \frac{4}{3}e^2 h^k_i\, F^p{}_k\, F_{pj}\, v^j \\
&\quad + \frac{4}{3}m\, e\, \dot{F}_{ij}\, v^j - 2\, G\, a_i + 2\, e^2 K(u)\, F_{ij}\, v^j + \hat{U}(u)\, c_i + \hat{V}(u)\, d_i + O_3 \ ,
\end{aligned} \tag{3.73}$$

where we have written $\hat{U} = (e^2 U(u) + 2\, m\, X(u))/2 = O_2$ and $\hat{V} = (e^2 V(u) + 2\, m\, Y(u))/2 = O_2$. Putting $A_1 = O_3$ and discarding the geometrically trivial

homogeneous solution of (3.72) we see that $Q_2 = -A_2/12$ and this describes smooth non–trivial perturbations of the wave fronts near the black hole as required. $G(u)$ is given by (3.60) with $A_0 = O_3$ and thus can be written as a definite integral for which we would naturally choose $(-\infty, u]$ as the range of integration resulting in $-2\,G\,a_i$ in (3.73) contributing to a 'tail term'. We can write $\hat{U}\,c_i + \hat{V}\,d_i = \Omega_{ij}\,v^j$ with $\Omega_{ij} = -\Omega_{ji}$ vanishing except for $\Omega_{13} = \Omega_{41} = \hat{U}/2$ and $\Omega_{23} = \Omega_{42} = \hat{V}/2$. Define $\omega_{ij}(u) = -\omega_{ji}(u)$ by $\dot{\omega}_{ij} = 2\,e^2 K(u)\,F_{ij} + \Omega_{ij}$. If we now make the 1–parameter family of infinitesimal Lorentz transformations on the unit tangent vector, $v^i \rightarrow \bar{v}^i = v^i - (4/3)m\,e\,F^i{}_j\,v^j - (1/m)\omega^i{}_j\,v^j$, we find that, after dropping the bar on v and its derivatives with respect to u, the equations of motion take the form

$$m\,a_i = e\,F_{ij}\,v^j + \frac{2}{3}e^2 h_i^k \dot{a}_k + \frac{4}{3}e^2 h_i^k\,F^p{}_k\,F_{pj}\,v^j + \mathcal{T}_i + O_3 \ , \tag{3.74}$$

where

$$\mathcal{T}_i = \frac{e}{m}\left\{\omega_k{}^j\,F_{ji} - \omega_i{}^j\,F_{jk} - 2\,G\,F_{ik}\right\} v^k = O_2 \ . \tag{3.75}$$

Since ω_{ij} can be written as an integral, whose range we would naturally take to be $(-\infty, u]$ as in the case of G above, we see that (3.75) is a 'tail term' in a set of equations of motion of the De Witt/ Brehme [1] form. Direct mathematical comparison with the equations of motion in [1] is highly non–trivial since the equations obtained in [1] have involved the removal of an infinite term, while (3.74) does not involve such a procedure. Hence we say that (3.74) fits the pattern of the equations obtained in [1].

4 Discussion

Perhaps the most significant aspect of the work presented in this paper is the fact that the equations of motion (3.74) of a small charged black hole moving in an external electromagnetic and gravitational field have been derived from the vacuum Einstein–Maxwell field equations without encountering infinities. It is also important to note that the external fields have been introduced as a solution of the Einstein–Maxwell vacuum field equations.

The second term on the right hand side of (3.74) is the electromagnetic radiation reaction 4–force. The third term on the right hand side of (3.74) can be written in terms of the electromagnetic energy tensor $E_{ij}(u)$ of the background space–time, calculated on $r = 0$ in coordinates (X^i). Using the background field equations on $r = 0$, $R_{ij} = 2\,E_{ij}$, this term can be written in terms of the background Ricci tensor components R_{ij} as $(2/3)e^2 h_i^k\,R_{kj}\,v^j$. This is the 4–force due to the external field and it differs from that derived

by Hobbs [35] by a factor of 2. The external electromagnetic field in [35] is *not* required to satisfy the Einstein–Maxwell vacuum field equations and so the background Ricci tensor appearing in [35] does not have the external electromagnetic field as source. Thus in effect half of the contribution to the external 4–force is neglected.

The 'tail term' in the form (3.75) vanishes if the external electromagnetic field vanishes. It involves three arbitrary functions (of u) $K, \hat{U}, \hat{V}$ with the latter two obtained from U, V, X, Y as indicated following (3.73). The mathematical origin of these functions as functions of integration arising in the determination of $\hat{K}_{-1}, l_2$ and m_2, required for the perturbed 4–potential, and of $\hat{a}_0$ and $\hat{b}_0$ required for the perturbed metric tensor, is clear from this work. In addition the tail term indirectly involves the two vector fields c^i and d^i orthogonal to v^i. It is a topic for further study to understand physically the role of these arbitrary functions and of these unique space–like vectors. Had the tail term not been so explicit we could not identify these arbitrary functions and space–like vectors for further consideration. By comparison the tail term obtained in [1] is an integral whose integrand is expressed in terms of functions appearing in the Hadamard form of the Green function of the vector wave equation. Very little is known explicitly about the form of such functions. The De Witt-Brehme tail term has proved quite intractable to analyse due to the fact that the Green function is not known in closed form. This makes a comparison with our result extremely difficult. Such a study would surely merit a paper independently of ours.

The original calculations of Dirac [2] were carried out in Minkowskian space–time. Our results do not specialise to those of Dirac since our background space–time is an Einstein–Maxwell space–time and if it is flat then the external electromagnetic field vanishes. In this case (3.74) gives $m\, a_i = O_2$ and when this is substituted into the surviving radiation reaction 4–force the latter is absorbed into the O_3–error. Hence we obtain geodesic motion (and in particular no "run–away" motion) at this level of approximation in this theory.

Acknowledgement

We are grateful to Takashi Fukumoto for carrying out helpful algebraic computer calculations in the early stages of this work. P.A.H. thanks the JSPS for a Long-Term Invitation Fellowship. This work is partly supported by a Grant-in-Aid for the G-COE Program "Weaving Science Web Beyond Particle–Matter Hierarchy" in Tohoku University, funded by the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. Y.I.

is in part supported by a Grant-in-Aid for Young Scientists (B)(20740118) from MEXT.

References

[1] B. S. De Witt and R. W. Brehme, Ann. Phys. NY **9**, 220 (1960).

[2] P. A. M. Dirac, Proc. R. Soc. A**167**, 148 (1932).

[3] R. Beig, Acta Phys. Asut. **38**, 300 (1973).

[4] A. O. Barut and D. Villaroel, J. Phys. A: Math. Gen. **8**, L1537 (1975).

[5] Y. Mino, M. Sasaki and T. Tanaka, Phys. Rev. D**55**, 3457 (1997).

[6] T. C. Quinn and R. M. Wald, Phys. Rev. D**56**, 3381 (1997).

[7] S. Detweiler and B. F. Whiting, Phys. Rev. D**67**, 024025 (2003).

[8] E. Poisson, Class. and Quantum Grav. **21**, R153 (2004).

[9] E. Poisson, Living Rev. Rel. **7**, 6 (2004).

[10] S. E. Gralla and R. M. Wald, Class. and Quantum Grav. **25**, 205009 (2008).

[11] L. Barack, Y. Mino, H. Nakano, A. Ori and M. Sasaki, Phys. Rev. Lett. **88**, 091101 (2002).

[12] L. Barack and A. Ori, Phys. Rev. D**66**, 084022 (2002).

[13] L. Barack and A. Ori, Phys. Rev. D**67**, 024029 (2003).

[14] L. Barack and A. Ori, Phys. Rev. Lett. **90**, 111101 (2002).

[15] Y. Mino, H. Nakano and M. Sasaki, Prog. Theor. Phys. **108**, 1039 (2003).

[16] S. Detweiler, E. Messaritaki and B. F. Whiting, Phys. Rev. D**67**, 104016 (2003).

[17] L. Barack and C. O. Lousto, Phys. Rev. D**66**, 061502 (2002).

[18] M. J. Pfenning and E. Poisson, Phys. Rev. D**65**, 084001 (2002).

[19] P. A. Hogan and I. Robinson, Found. Phys. **15**, 617 (1985).

[20] E. T. Newman and R. Posadas, Phys. Rev. **187**, 1784 (1969).

[21] I. Robinson and J. R. Robinson, in *General Relativity: papers in honour of J. L. Synge*, ed. L. O'Raifeartaigh (Clarendon Press, Oxford 1972), p.151.

[22] P. A. Hogan and M. Imaeda, J. Phys. A: Math. Gen. **12**, 1061 (1979).

[23] P. A. Hogan and A. Trautman, in *Gravitation and Geometry*, edited by W. Rindler and A. Trautman (Bibliopolis, Naples, 1987), p. 215.

[24] T. Futamase and P. A. Hogan, J. Math. Phys. **34**, 154 (1993).

[25] R. K. Sachs, Proc. R. Soc. London, Ser. A**265**, 463 (1962).

[26] E. T. Newman and T. W. J. Unti, J. Math. Phys. **3**, 891 (1962).

[27] I. Robinson and A. Trautman, Phys. Rev. Lett. **4**, 431 (1960).

[28] I. Robinson and A. Trautman, Proc. R. Soc. London, Ser. A**270**, 463 (1962).

[29] E. Fermi, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat.Nat. **31**, 21 and 51 (1922).

[30] L. O'Raifeartaigh, Proc. R. Irish Acad. Ser. A**59**, 15 (1958).

[31] L. O'Raifeartaigh, Proc. R. Irish Acad. Ser. A**62**, 63 (1962).

[32] E. T. Newman and T. W. J. Unti, J. Math. Phys. **4**, 1467 (1969).

[33] T. Molenda, "Contributed Papers, vol.1, 10th. International Conference on General Relativity and Gravitation 1983"(B. Bertotti, F. de Felice and A. Pascolini, Eds.), p.97.

[34] P. A. Hogan and G. F. R. Ellis, Ann. Phys. NY **195**, 293 (1989).

[35] J. M. Hobbs, Ann. Phys. NY **47**, 141 (1968).

A Useful Formulas from Minkowskian Geometry

We record here some useful equations which hold exactly in Minkowskian space–time and are also important in the neighbourhood of the world line $r = 0$ in the background space–time in this paper. The transformation (2.24) can in principle be inverted giving (x, y, r, u) as functions of (X^i) which means that we can consider (x, y, r, u) as scalar fields on Minkowskian space–time. Their derivatives with respect to (X^i), denoted by a comma, are obtained by first differentiating (2.24) to arrive at

$$\delta^i_j = (v^i - r\,h_0\,k^i)\,u_{,j} + k^i\,r_{,j} + r\,\frac{\partial k^i}{\partial x}\,x_{,j} + r\,\frac{\partial k^i}{\partial y}\,y_{,j}\ . \qquad \text{(A.1)}$$

Multiplying this by k_i gives immediately

$$k_j = u_{,j}\ . \qquad \text{(A.2)}$$

Now multiplying (A.1) by v_j yields

$$r_{,j} = v_j - (1 - r\,h_0)\,k_j\ . \qquad \text{(A.3)}$$

Differentiating (2.24) with respect to X^j and using (A.2) and (A.3) results in the alternative form for (A.1):

$$k^i{}_{,j} = \frac{1}{r}\left(\delta^i_j - v^i\,k_j - k^i\,v_j + (1 - r\,h_0)\,k^i\,k_j\right)\ , \qquad \text{(A.4)}$$

with $h_0 = a_i\,k^i$. From this it follows that

$$\frac{\partial k^i}{\partial r} = k^i{}_{,j}\,k^j = 0\ , \qquad \frac{\partial k^i}{\partial u} = k^i{}_{,j}\,v^j = -h_0\,k^i\ . \qquad \text{(A.5)}$$

Writing $h_0 = \partial(\log P_0)/\partial u$ for some function P_0 independent of r we see from the latter two equations that $k^i = P_0^{-1}\,\zeta^i$ with ζ^i null and independent of r and u. Thus ζ^i can be parametrized by two real parameters. We have chosen the parameters x, y as given in (2.23) and then $P_0 = \zeta_i\,v^i$ in (2.20) is a consequence of this choice. A more complete discussion of this construction can be found in [33] and [34].

We can now simplify (A.1) to read

$$\delta^i_j = v^i\,k_j + k^i\,v_j - k^i\,k_j + r\,\frac{\partial k^i}{\partial x}\,x_{,j} + r\,\frac{\partial k^i}{\partial y}\,y_{,j}\ . \qquad \text{(A.6)}$$

Multiplying (A.6) by $\partial k_i/\partial x$ and also by $\partial k_i/\partial y$ provides the remaining equations needed:

$$x_{,j} = -\frac{1}{r}P_0^2\,\frac{\partial k_j}{\partial x}\ , \qquad y_{,j} = -\frac{1}{r}P_0^2\,\frac{\partial k_j}{\partial y}\ . \tag{A.7}$$

Substituting (A.7) into (A.6) results in the Minkowskian metric tensor being written in the form

$$\eta^{ij} = -P_0^2\left(\frac{\partial k^i}{\partial x}\,\frac{\partial k^j}{\partial x} + \frac{\partial k^i}{\partial y}\,\frac{\partial k^j}{\partial y}\right) + k^i\,v^j + k^j\,v^i - k^i\,k^j\ , \tag{A.8}$$

and much use is made of this relation in the calculations behind this paper. Equivalently the Minkowskian line–element takes the form

$$ds^2 = \eta_{ij}\,dX^i\,dX^j = -r^2P_0^{-2}(dx^2+dy^2) + 2\,du\,dr + (1-2\,h_0\,r)\,du^2\ . \tag{A.9}$$

This suggests we introduce basis 1–forms (defining a half null basis)

$$\vartheta^1 = r\,P_0^{-1}dx\ ,\ \vartheta^2 = r\,P_0^{-1}dy\ ,\ \vartheta^3 = dr+\frac{1}{2}(1-2\,h_0\,r)\,du\ ,\ \vartheta^4 = du\ . \tag{A.10}$$

Using (A.2), (A.3) and (A.7) we can express these in terms of the rectangular Cartesian coordinates and time as

$$\vartheta^1 = -P_0\,\frac{\partial k_i}{\partial x}\,dX^i = -\vartheta_1\ , \tag{A.11}$$

$$\vartheta^2 = -P_0\,\frac{\partial k_i}{\partial y}\,dX^i = -\vartheta_2\ , \tag{A.12}$$

$$\vartheta^3 = (v_i - \frac{1}{2}k_i)\,dX^i = \vartheta_4\ , \tag{A.13}$$

$$\vartheta^4 = k_i\,dX^i = \vartheta_3\ . \tag{A.14}$$

It is helpful to have available the second partial derivatives of k^i with respect to x and y expressed on the basis consisting of the vectors $v^i,\ k^i,\ \partial k^i/\partial x$ and $\partial k^i/\partial y$. These formulas are:

$$\frac{\partial^2 k^i}{\partial x^2} = P_0^{-2}(v^i-k^i) - \frac{\partial}{\partial x}(\log P_0)\,\frac{\partial k^i}{\partial x} + \frac{\partial}{\partial y}(\log P_0)\,\frac{\partial k^i}{\partial y}\ , \tag{A.15}$$

$$\frac{\partial^2 k^i}{\partial y^2} = P_0^{-2}(v^i-k^i) + \frac{\partial}{\partial x}(\log P_0)\,\frac{\partial k^i}{\partial x} - \frac{\partial}{\partial y}(\log P_0)\,\frac{\partial k^i}{\partial y}\ , \tag{A.16}$$

$$\frac{\partial^2 k^i}{\partial x\partial y} = -\frac{\partial}{\partial y}(\log P_0)\,\frac{\partial k^i}{\partial x} - \frac{\partial}{\partial x}(\log P_0)\,\frac{\partial k^i}{\partial y}\ . \tag{A.17}$$

B Perturbed Field Equations

Writing, for the perturbed space–time described in section 3, $\mathcal{M}^a = \hat{F}^{ab}{}_{|b}$ and $\mathcal{E}_{ab} = \hat{R}_{ab} - 2\,\hat{E}_{ab}$ the leading terms and the errors we tolerate in these expressions (and thus the extent to which we satisfy the vacuum Einstein–Maxwell field equations) are given here. If the equations of motion are required with greater accuracy then the field equations have to be solved with greater accuracy resulting in smaller errors in the coefficients of these powers of r.

$$\begin{aligned}
\mathcal{M}^A &= O_3\times r^{-2} + O_2\times r^{-1} + O_1 + O(r)\ ,\ (A=1,2), && \text{(B.1)}\\
\mathcal{M}^3 &= O_3\times r^{-1} + O_1 + O(r)\ , && \text{(B.2)}\\
\mathcal{M}^4 &= O_2\times r + O(r^2)\ , && \text{(B.3)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{AA} &= O_3\times r^{-4} + O_3\times r^{-2} + O_2\times r^{-1} + O(r^0)\ , && \text{(B.4)}\\
\mathcal{E}_{11} - \mathcal{E}_{22} &= O_3\times r^{-2} + O_2\times r^{-1} + O_1 + O(r)\ , && \text{(B.5)}\\
\mathcal{E}_{12} &= O_3\times r^{-2} + O_2\times r^{-1} + O_1 + O(r)\ , && \text{(B.6)}\\
\mathcal{E}_{A3} &= O_3\times r^{-2} + O_1\times r^{-1} + O_1 + O(r)\ , && \text{(B.7)}\\
\mathcal{E}_{A4} &= O_2\times r^{-3} + O_2\times r^{-2} + O_2\times r^{-1} + O(r^0)\ , && \text{(B.8)}\\
\mathcal{E}_{33} &= O_1 + O(r)\ , && \text{(B.9)}\\
\mathcal{E}_{34} &= O_4\times r^{-4} + O_3\times r^{-3} + O_2\times r^{-2}\\
&\quad + O_2\times r^{-1} + O(r^0)\ , && \text{(B.10)}\\
\mathcal{E}_{44} &= O_3\times r^{-3} + O_3\times r^{-2} + O(r^{-1})\ . && \text{(B.11)}
\end{aligned}$$

The coefficients of the various powers of r in these expressions are calculated by substituting into $\mathcal{M}^a$ and $\mathcal{E}_{ab}$ the metric tensor, given via the line–element (2.1) with the expansions (3.1)–(3.6), and the potential 1–form (2.14), with the expansions (3.7)–(3.9). Writing (B.1)–(B.11) in the form

$$\mathcal{M}^a = \sum {}_{(n)}\mathcal{M}^a\, r^n\ , \qquad \mathcal{E}_{ab} = \sum {}_{(n)}\mathcal{E}_{ab}\, r^n\ , \tag{B.12}$$

the coefficients ${}_{(n)}\mathcal{M}^a$ and ${}_{(n)}\mathcal{E}_{ab}$ required to establish (B.1)–(B.11) are:

$$\begin{aligned}
{}_{(-2)}\mathcal{M}^1 &= \frac{\partial}{\partial y}\left\{\hat{P}_0^2\left(\frac{\partial \hat{M}_0}{\partial x} - \frac{\partial \hat{L}_0}{\partial y}\right)\right\} + 2e^2\hat{L}_2 - e\,\hat{a}_{-1}\,\hat{P}_0^{-2}\\
&\quad + \frac{\partial \hat{K}_{-1}}{\partial x} + O_3\ , && \text{(B.13)}\\
{}_{(-1)}\mathcal{M}^1 &= O_3\ , && \text{(B.14)}\\
{}_{(0)}\mathcal{M}^1 &= 6\,m\,L_3 + O_2 = O_1\ , && \text{(B.15)}
\end{aligned}$$

$$
\begin{aligned}
{}_{(-2)}\mathcal{M}^2 &= \frac{\partial}{\partial x}\left\{\hat{P}_0^2\left(\frac{\partial \hat{M}_0}{\partial x}-\frac{\partial \hat{L}_0}{\partial y}\right)\right\}-2e^2\hat{M}_2+\ e\,\hat{b}_{-1}\hat{P}_0^{-2} \\
&\quad -\frac{\partial \hat{K}_{-1}}{\partial y}+O_3\ , && \text{(B.16)}\\
{}_{(-1)}\mathcal{M}^2 &= O_3\ , && \text{(B.17)}\\
{}_{(0)}\mathcal{M}^2 &= -6\,m\,M_3+O_2=O_1\ , && \text{(B.18)}\\
{}_{(-1)}\mathcal{M}^3 &= O_2\ , && \text{(B.19)}\\
{}_{(0)}\mathcal{M}^3 &= -4mP_0^{-2}F_{ij}k^i\,v^j+O_2=O_1\ , && \text{(B.20)}\\
{}_{(1)}\mathcal{M}^4 &= -2\hat{P}_0^{-2}\hat{K}_1+4\hat{P}_0^{-2}\left(\hat{a}_{-1}\hat{L}_2+\hat{b}_{-1}\hat{M}_2-e\,\hat{q}_2\right)\\
&\quad +2\left(\frac{\partial \hat{L}_2}{\partial x}+\frac{\partial \hat{M}_2}{\partial y}\right)+O_2\ . && \text{(B.21)}
\end{aligned}
$$

Writing

$$
\hat{\Delta}=\hat{P}_0^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\ , \qquad \text{(B.22)}
$$

we find that

$$
\begin{aligned}
{}_{(-4)}\mathcal{E}_{AA} &= O_3\ , && \text{(B.23)}\\
{}_{(-3)}\mathcal{E}_{AA} &= O_3\ , && \text{(B.24)}\\
{}_{(-2)}\mathcal{E}_{AA} &= 2\,\hat{\Delta}\log\hat{P}_0-4\,\hat{P}_0^4\left(\frac{\partial \hat{M}_0}{\partial x}-\frac{\partial \hat{L}_0}{\partial y}\right)\left(\frac{\partial \hat{M}_2}{\partial x}-\frac{\partial \hat{L}_2}{\partial y}\right)\\
&\quad -3\hat{P}_0^2\left(\frac{\partial}{\partial x}(\hat{P}_0^{-2}\hat{a}_{-1})+\frac{\partial}{\partial y}(\hat{P}_0^{-2}\hat{b}_{-1})\right)\\
&\quad +4\,e\hat{K}_1+4\,\hat{K}_1\,\hat{K}_{-1}-8\,e(\hat{a}_{-1}\hat{L}_2+\hat{b}_{-1}\hat{M}_2)\\
&\quad -2\,\hat{c}_0+12\,e^2\hat{q}_2-\frac{1}{2}\hat{P}_0^{-2}(\hat{a}_{-1}^2+\hat{b}_{-1}^2)+O_3\ , && \text{(B.25)}\\
{}_{(-1)}\mathcal{E}_{AA} &= -4\,\hat{c}_1-8\,\frac{\partial\log\hat{P}_0}{\partial u}+8\,e\,\hat{K}_2-32\,m\,\hat{q}_2\\
&\quad -4\,\hat{P}_0^2\left(\frac{\partial}{\partial x}(\hat{P}_0^{-2}\hat{a}_0)+\frac{\partial}{\partial y}(\hat{P}_0^{-2}\hat{b}_0)\right)+O_2\ , && \text{(B.26)}\\
{}_{(-2)}\mathcal{E}_{11}-{}_{(-2)}\mathcal{E}_{22} &= \frac{\partial \hat{b}_{-1}}{\partial y}-\frac{\partial \hat{a}_{-1}}{\partial x}-4\,e^2\hat{\alpha}_2-\frac{1}{2}\hat{P}_0^{-2}(\hat{a}_{-1}^2+\hat{b}_{-1}^2)\\
&\quad -8\,e^2\hat{P}_0^2(\hat{L}_2^2-\hat{M}_2^2)-8\,\hat{P}_0^2\hat{L}_2\,\frac{\partial \hat{K}_{-1}}{\partial x}+8\,\hat{P}_0^2\hat{M}_2\,\frac{\partial \hat{K}_{-1}}{\partial y}\\
&\quad +O_3\ , && \text{(B.27)}\\
{}_{(-1)}\mathcal{E}_{11}-{}_{(-1)}\mathcal{E}_{22} &= 16\,m\,\hat{P}_0^2(\hat{L}_2^2-\hat{M}_2^2)-8\,\hat{P}_0^2\hat{L}_2\,\frac{\partial \hat{K}_0}{\partial x}+8\,\hat{P}_0^2\hat{M}_2\,\frac{\partial \hat{K}_0}{\partial y}
\end{aligned}
$$

$$+16\,m\,\hat{\alpha}_2 - 2\,\frac{\partial \hat{a}_0}{\partial x} + 2\,\frac{\partial \hat{b}_0}{\partial y} + O_2 \ , \tag{B.28}$$

$$\begin{aligned}
{}_{(0)}\mathcal{E}_{11} - {}_{(0)}\mathcal{E}_{22} &= -8\hat{c}_0\,\hat{P}_0^2(\hat{L}_2^2 - \hat{M}_2^2) - 8\,\hat{P}_0^2\hat{L}_2\,\frac{\partial \hat{K}_1}{\partial x} + 8\,\hat{P}_0^2\hat{M}_2\,\frac{\partial \hat{K}_1}{\partial y} \\
&\quad -12\,\hat{\alpha}_2\,\hat{c}_0 - 3\frac{\partial \hat{a}_1}{\partial x} + 3\frac{\partial \hat{b}_1}{\partial y} + O_1 = O_1 \ , \qquad \text{(B.29)}
\end{aligned}$$

$$\begin{aligned}
{}_{(-2)}\mathcal{E}_{12} &= -\frac{1}{2}\frac{\partial \hat{a}_{-1}}{\partial y} - \frac{1}{2}\frac{\partial \hat{b}_{-1}}{\partial x} - 4\,e^2\hat{\beta}_2 - \frac{1}{2}\hat{P}_0^{-2}\hat{a}_{-1}\,\hat{b}_{-1} \\
&\quad -8\,e^2\hat{P}_0^2\hat{L}_2\hat{M}_2 - 4\,\hat{P}_0^2\hat{L}_2\,\frac{\partial \hat{K}_{-1}}{\partial y} - 4\,\hat{P}_0^2\hat{M}_2\,\frac{\partial \hat{K}_{-1}}{\partial x} \\
&\quad +O_3 \ , \qquad \text{(B.30)}
\end{aligned}$$

$$\begin{aligned}
{}_{(-1)}\mathcal{E}_{12} &= 16\,m\,\hat{P}_0^2\hat{L}_2\,\hat{M}_2 - 4\,\hat{P}_0^2\hat{L}_2\,\frac{\partial \hat{K}_0}{\partial y} - 4\,\hat{P}_0^2\hat{M}_2\,\frac{\partial \hat{K}_0}{\partial x} \\
&\quad +8\,m\,\hat{\beta}_2 - \frac{\partial \hat{a}_0}{\partial y} - \frac{\partial \hat{b}_0}{\partial x} + O_2 \ , \qquad \text{(B.31)}
\end{aligned}$$

$$\begin{aligned}
{}_{(0)}\mathcal{E}_{12} &= -8\hat{c}_0\,\hat{P}_0^2\hat{L}_2\,\hat{M}_2 - 4\,\hat{P}_0^2\hat{M}_2\,\frac{\partial \hat{K}_1}{\partial x} - 4\,\hat{P}_0^2\hat{L}_2\,\frac{\partial \hat{K}_1}{\partial y} \\
&\quad -6\,\hat{\beta}_2\,\hat{c}_0 - \frac{3}{2}\frac{\partial \hat{a}_1}{\partial y} - \frac{3}{2}\frac{\partial \hat{b}_1}{\partial x} + O_1 = O_1 \ , \qquad \text{(B.32)}
\end{aligned}$$

$$\begin{aligned}
{}_{(-2)}\mathcal{E}_{13} &= 4\,\hat{M}_2\,\hat{P}_0^3\left(\frac{\partial \hat{L}_0}{\partial y} - \frac{\partial \hat{M}_0}{\partial x}\right) + 4\,e\,\hat{L}_2\hat{P}_0 + \hat{P}_0^{-1}\hat{a}_{-1} \\
&\quad +4\,\hat{K}_{-1}\,\hat{L}_2\,\hat{P}_0 + O_3 \ , \qquad \text{(B.33)}
\end{aligned}$$

$${}_{(-1)}\mathcal{E}_{13} = 6\,e\,P_0\,L_3 + O_2 = O_1 \ , \tag{B.34}$$

$$\begin{aligned}
{}_{(0)}\mathcal{E}_{13} &= -4\,\hat{K}_1\,\hat{L}_2\,\hat{P}_0 + 2\,\hat{P}_0^3\left(\frac{\partial}{\partial x}(\hat{P}_0^{-2}\hat{\alpha}_2) + \frac{\partial}{\partial y}(\hat{P}_0^{-2}\hat{\beta}_2)\right) \\
&\quad +2\,\hat{P}_0\,\frac{\partial \hat{q}_2}{\partial x} - 2\,\hat{P}_0^{-1}\hat{a}_1 + 4\,\hat{P}_0^3\hat{M}_2\,\left(\frac{\partial \hat{L}_2}{\partial y} - \frac{\partial \hat{M}_2}{\partial x}\right) + O_1 \\
&\quad = O_1 \ , \qquad \text{(B.35)}
\end{aligned}$$

$${}_{(-3)}\mathcal{E}_{14} = 4\,m\,e\,P_0\,F_{ij}\,\frac{\partial k^i}{\partial x}\,v^j + O_3 = O_2 \ , \tag{B.36}$$

$$\begin{aligned}
{}_{(-2)}\mathcal{E}_{14} &= 2\,e\,\hat{c}_0\hat{L}_2\,\hat{P}_0 + 2\,e\,\hat{P}_0\frac{\partial \hat{K}_1}{\partial x} + \frac{1}{2}\,\hat{P}_0^{-1}\hat{a}_{-1}\,\hat{c}_0 \\
&\quad -\frac{1}{2}\hat{P}_0\frac{\partial}{\partial y}\left(\hat{P}_0^2\left\{\frac{\partial}{\partial y}(\hat{P}_0^{-2}\hat{a}_{-1}) - \frac{\partial}{\partial x}(\hat{P}_0^{-2}\hat{b}_{-1})\right\}\right) \\
&\quad -\hat{P}_0^{-1}\hat{a}_{-1}\,\hat{\Delta}\log\hat{P}_0 + O_2 = O_2 \ , \qquad \text{(B.37)}
\end{aligned}$$

$$
\begin{aligned}
{}_{(-1)}\mathcal{E}_{14} &= O_2\ , &\text{(B.38)}\\
{}_{(-2)}\mathcal{E}_{23} &= -4\,\hat{L}_2\,\hat{P}_0^3\left(\frac{\partial\hat{L}_0}{\partial y}-\frac{\partial\hat{M}_0}{\partial x}\right)+4\,e\,\hat{M}_2\hat{P}_0+\hat{P}_0^{-1}\hat{b}_{-1}\\
&\quad +4\,\hat{K}_{-1}\,\hat{M}_2\,\hat{P}_0+O_3\ , &\text{(B.39)}\\
{}_{(-1)}\mathcal{E}_{23} &= 6\,e\,P_0\,M_3+O_2=O_1\ , &\text{(B.40)}\\
{}_{(0)}\mathcal{E}_{23} &= -4\,\hat{K}_1\,\hat{M}_2\,\hat{P}_0+2\,\hat{P}_0^3\left(-\frac{\partial}{\partial y}(\hat{P}_0^{-2}\hat{\alpha}_2)+\frac{\partial}{\partial x}(\hat{P}_0^{-2}\hat{\beta}_2)\right)\\
&\quad +2\,\hat{P}_0\,\frac{\partial\hat{q}_2}{\partial y}-2\,\hat{P}_0^{-1}\hat{b}_1-4\,\hat{P}_0^3\hat{L}_2\,\left(\frac{\partial\hat{L}_2}{\partial y}-\frac{\partial\hat{M}_2}{\partial x}\right)+O_1\\
&\quad =O_1\ , &\text{(B.41)}\\
{}_{(-3)}\mathcal{E}_{24} &= 4\,m\,e\,P_0\,F_{ij}\frac{\partial k^i}{\partial y}\,v^j+O_3=O_2\ , &\text{(B.42)}\\
{}_{(-2)}\mathcal{E}_{24} &= 2\,e\,\hat{c}_0\hat{M}_2\,\hat{P}_0+2\,e\,\hat{P}_0\frac{\partial\hat{K}_1}{\partial y}+\frac{1}{2}\,\hat{P}_0^{-1}\hat{b}_{-1}\,\hat{c}_0\\
&\quad +\frac{1}{2}\hat{P}_0\frac{\partial}{\partial x}\left(\hat{P}_0^2\frac{\partial}{\partial y}(\hat{P}_0^{-2}\hat{a}_{-1})-\hat{P}_0^2\frac{\partial}{\partial x}(\hat{P}_0^{-2}\hat{b}_{-1})\right)\\
&\quad -\hat{P}_0^{-1}\hat{b}_{-1}\,\hat{\Delta}\log\hat{P}_0+O_2=O_2\ , &\text{(B.43)}\\
{}_{(-1)}\mathcal{E}_{24} &= O_2\ , &\text{(B.44)}\\
{}_{(0)}\mathcal{E}_{33} &= -8\,\hat{P}_0^2(\hat{L}_2^2+\hat{M}_2^2)+12\,\hat{q}_2+O_3\ , &\text{(B.45)}\\
{}_{(-4)}\mathcal{E}_{34} &= O_4\ , &\text{(B.46)}\\
{}_{(-3)}\mathcal{E}_{34} &= O_3\ , &\text{(B.47)}\\
{}_{(-2)}\mathcal{E}_{34} &= 2\,e\,\hat{K}_1+\frac{1}{2}\hat{P}_0^2\left(\frac{\partial}{\partial x}(\hat{P}_0^{-2}\hat{a}_{-1})+\frac{\partial}{\partial y}(\hat{P}_0^{-2}\hat{b}_{-1})\right)\\
&\quad +O_2\ , &\text{(B.48)}\\
{}_{(-1)}\mathcal{E}_{34} &= \hat{c}_1+2\,\hat{P}_0^{-1}\frac{\partial\hat{P}_0}{\partial u}+4\,e\,\hat{K}_2+8\,m\,\hat{q}_2\\
&\quad +\hat{P}_0^2\left(\frac{\partial}{\partial x}(\hat{P}_0^{-2}\hat{a}_0)+\frac{\partial}{\partial y}(\hat{P}_0^{-2}\hat{b}_0)\right)+O_2\ , &\text{(B.49)}\\
{}_{(-4)}\mathcal{E}_{44} &= O_3\ , &\text{(B.50)}\\
{}_{(-3)}\mathcal{E}_{44} &= 2\,m\,\hat{P}_0^2\left(\frac{\partial}{\partial x}(\hat{P}_0^{-2}\hat{a}_{-1})+\frac{\partial}{\partial y}(\hat{P}_0^{-2}\hat{b}_{-1})\right)\\
&\quad -4\,e^2\hat{P}_0\,\frac{\partial\hat{P}_0}{\partial u}-2\,\hat{\Delta}\hat{f}_{-1}+O_3 &\text{(B.51)}\\
{}_{(-2)}\mathcal{E}_{44} &= T_1+T_2+T_3+O_3\ . &\text{(B.52)}
\end{aligned}
$$

In the final equation here the terms $T_1,\ T_2,\ T_3$ are given in the text by (3.44), (3.55) and (3.57) respectively.